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NOTE ON ASYMPTOTICAL BEHAVIOR OF SOLUTIONS OF EMDEN-FOWLER EQUATION AND THE EXISTENCE AND UNIQUENESS OF SOLUTION OF SOME CAUCHY PROBLEM

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Abstract. We study differential equation of Emden-Fowler type

$$y'' - x^a y^\sigma = 0, \quad \text{for } a \in \mathbb{R} \text{ and } \sigma < 0.$$

We describe the conditions on parameters a and σ which assure that this equation has infinitely many solutions defined in some neighborhood of zero and the conditions which guarantee existence of infinitely solutions with certain asymptotic behavior.

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1. Introduction

In this paper we study the very important non-linear second-order differential equation

$$y'' - x^a y^\sigma = 0.$$

This equation came first into prominence in connection with the astrophysical researcher Emden. A number of results obtained by Emden in the usual half-intuitive, wholly ingenious fashion of the physicist were made by Fowler, who was then stimulated to continue and give a complete discussion of solutions of this equation for all values of the parameters. The equation has several very interesting physical applications, occurring in astrophysics in the form of the Emden equation and in atomic physics in the form of Fermi-Thomas equation.

Mathematically, the equation has great potential. It is a nontrivial, nonlinear, differential equation with a large class of solutions whose behavior can be ascertained with astonishing accuracy, despite the fact that the solutions, in general, can't be obtained explicitly. The Emden-Fowler type of equation has significant applications in

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many fields of scientific and technical world and this equation has been investigated by many researchers. Such equations were considered, e.g. in [1–5]. In [1,2] asymptotic properties of solutions of the Emden-Fowler equations at infinity were obtained. In papers [3–5] some asymptotic properties of solutions of the Emden-Fowler equations near zero were obtained. These papers motivate us to fully examine the behavior of solutions Emden-Fowler equation near zero in the case when $\sigma < 0$ and $a \in \mathbb{R}$.

Our paper is divided into two parts. In Section 2, we discuss conditions on parameters which ensure that the solution of Emden-Fowler equation is defined in some neighborhood of zero and it is shown that this result can not be proven under weaker assumptions.

In Section 3, we examine the asymptotic behavior of the solutions of this equation. We proved the existence of infinitely many solutions of this equation for some value of parameters. Also, we proved uniqueness of solution which satisfies appropriate conditions.

With some changes of the variables, we can reduce equation

$$(t^{\rho} \cdot u'(t))' - t^{\alpha} \cdot u^{\sigma}(t) = 0, \quad \text{for } \rho \neq 1, \tag{1.1}$$

to the equation

$$y'' - x^{a} y^{\sigma} = 0. \tag{1.2}$$

Our results give appropriate results for equation (1.1). Namely, if $\rho > 1$, set $x = \dfrac{t^{\rho-1}}{\rho-1}$, $u = \dfrac{y}{x} \cdot (\rho-1)^{\frac{\rho-\alpha-2}{(\rho-1)(\sigma-1)}}$, the equation for y is $y''(x) - x^{a} y^{\sigma} = 0$, where $a = \dfrac{\alpha+\rho}{\rho-1} - (\sigma+3)$. If $\rho < 1$, set $x = \dfrac{t^{1-\rho}}{1-\rho}$, $u = y \cdot (1-\rho)^{-\frac{\rho+\alpha}{(1-\rho)(\sigma-1)}}$, the equation for y is $y''(x) - x^{a} y^{\sigma} = 0$, where $a = \dfrac{\alpha+\rho}{1-\rho}$.

2. Asymptotic behavior of solutions of Emden-Fowler equation

Consider the equation

$$y'' - x^{a} y^{\sigma} = 0, \tag{2.1}$$

where $\sigma < 0$ and $a \in \mathbb{R}$.

Using Picard theorem we can conclude existence of the local defined solution, so the natural question is to find conditions which assure that the solution is defined globally. First we will prove that for some values of parameters a and σ every monotone solution of equation (2.1) is defined on the interval containing zero environment, i.e. that there are no monotone solutions such that the graph of the solution "tends to the point on x–axis".

Lemma 1. *If $\sigma \leqslant -1$ and $x_1 > 0$, then every positive monotone solution of differential equation (2.1) is defined on some interval I such that $(0, x_1] \subset I$, $x_1 > 0$.*

Proof. It is enough to observe only the nondecreasing solutions. Let observe the solution such that $y(x_1) > 0$, $y'(x_1) > 0$. Suppose that $x_0 > 0$ and $y(x)$ is the solution defined on the interval $(x_0, x_1]$, but $y(x)$ is not defined on the interval $(x', x_0]$, for arbitrarily $x' < x_0$. In this situation we have $\lim\limits_{x\to x_0+} y(x) = 0$. It follows from equation (2.1) that $y''(x) > 0$, and from that we can conclude that $y'(x)$ is an increasing function, so there exists finite $\lim\limits_{x\to x_0+} y'(x)$, so we have $y(x) \leqslant y(x_1)$ and $y'(x) \leqslant y'(x_1)$ for every $x \in (x_0, x_1]$. The function $y(x)$ can be continuously extended in point x_0 such that $y(x_0) = 0$, then for every $x \in (x_0, x_1)$ and some $\xi \in (x_0, x)$ the inequality $\frac{y(x)-0}{x-x_0} = y'(\xi) \leqslant y'(x_1)$ holds. Since $\sigma < 0$, it follows that

$$y''(x) = x^a y^\sigma(x) \geqslant \big(y'(x_1)\big)^\sigma x^a (x-x_0)^\sigma,$$

and for every $x \in (x_0, x_1)$ we have

$$y'(x_1) \geqslant y'(x_1) - y'(x) = \int_x^{x_1} y''(t)\,dt \geqslant \big(y'(x_1)\big)^\sigma \int_x^{x_1} t^a (t-x_0)^\sigma\,dt.$$

Since $\sigma \leqslant -1$, the right side of the inequality tends to ∞ when $x \to x_0$, but the left side of the inequality is constant, which is a contradiction, so it follows that such solution is defined on the interval $(0, x_1]$. $\square$

Example 1. Let's

$$x(u) = x_0 \cdot e^{2\int\limits_0^u \frac{dv}{\sqrt{v^2+8\cdot\frac{v^{\sigma+1}}{\sigma+1}}}},\ y(u) = \sqrt{x_0}\cdot u \cdot e^{\int\limits_0^u \frac{dv}{\sqrt{v^2+8\cdot\frac{v^{\sigma+1}}{\sigma+1}}}},\ x_0 > 0,\ u \in (0,\infty),$$

where $a = -\dfrac{\sigma+3}{2}$ and $-1 < \sigma < 0$. We notice that $x(u)$ and $y(u)$ are well defined on the interval $(0,\infty)$. Also, $x(u)$ is an injective function from $(0,\infty)$ to (x_0,∞), so we can observe $x(u)$ and $y(u)$ as parametric representation of function $y(x)$, defined on (x_0,∞). Straightforward calculation shows that

$$x'(u) = \frac{2x_0 \cdot e^{2\int\limits_0^u \frac{dv}{\sqrt{v^2+8\cdot\frac{v^{\sigma+1}}{\sigma+1}}}}}{\sqrt{u^2+8\cdot\frac{u^{\sigma+1}}{\sigma+1}}},$$

$$y'(u) = \left(\sqrt{x_0} + \frac{u\cdot\sqrt{x_0}}{\sqrt{u^2+8\cdot\frac{u^{\sigma+1}}{\sigma+1}}}\right)\cdot e^{\int\limits_0^u \frac{dv}{\sqrt{v^2+8\cdot\frac{v^{\sigma+1}}{\sigma+1}}}},$$

$$x''(u) = \frac{2x_0}{u^2+8\cdot\frac{u^{\sigma+1}}{\sigma+1}}\cdot e^{2\int\limits_0^u \frac{dv}{\sqrt{v^2+8\cdot\frac{v^{\sigma+1}}{\sigma+1}}}}\cdot\left(2 - \frac{u+4u^\sigma}{\sqrt{u^2+8\cdot\frac{u^{\sigma+1}}{\sigma+1}}}\right),$$

$$y''(u)=\frac{\sqrt{x_0}\cdot e^{\int_0^u \frac{dv}{\sqrt{v^2+8\cdot\frac{v^{\sigma+1}}{\sigma+1}}}}}{\sqrt{u^2+8\cdot\frac{u^{\sigma+1}}{\sigma+1}}}\cdot\left(1+\frac{u}{\sqrt{u^2+8\cdot\frac{u^{\sigma+1}}{\sigma+1}}}+\frac{\frac{4(1-\sigma)}{1+\sigma}\cdot u^{\sigma+1}}{u^2+8\cdot\frac{u^{\sigma+1}}{\sigma+1}}\right).$$

It follows that

$$\begin{aligned}\frac{d^2y}{dx^2}(u)&=\frac{1}{(x'(u))^3}\left(y''(u)\cdot x'(u)-x''(u)\cdot y'(u)\right)\\ &=x_0^{-\frac{3}{2}}\cdot u^{\sigma}\cdot e^{-3\int_0^u \frac{dv}{\sqrt{v^2+8\cdot\frac{v^{\sigma+1}}{\sigma+1}}}},\ \text{and}\end{aligned}$$

$$(x(u))^a\cdot(y(u))^{\sigma}=(x(u))^{-\frac{\sigma+3}{2}}\cdot(y(u))^{\sigma}=x_0^{-\frac{3}{2}}\cdot u^{\sigma}\cdot e^{-3\int_0^u \frac{dv}{\sqrt{v^2+8\cdot\frac{v^{\sigma+1}}{\sigma+1}}}},$$

so with $x(u)$ and $y(u)$ a parametric representation of function $y(x)$ is given, which is a solution of equation (2.1). Notice that

$$x(u)\to x_0,\ y(u)\to 0,$$

when $u\to 0+$. It follows that the result of Lemma 1 can not be extended for every $-1<\sigma<0$ and $a+\sigma+1\leqslant 0$.

Lemma 1 provides existence of the solution in some neighborhood of zero, so the natural question is to investigate asymptotic behavior of such solutions. In the next theorem we show that for some values of parameters a and σ there are no solutions $y(x)$ of differential equation (2.1), such that $\lim\limits_{x\to 0+} y(x)=0$, i.e. that there are no solutions such that the graph of the solution ”tends to the point $(0,0)$”.

Theorem 1. *If $\sigma<0$ and $a+\sigma+1\leqslant 0$, then there are no positive solutions $y(x)$ of differential equation (2.1) defined in the some neighborhood of zero such that $\lim\limits_{x\to 0+} y(x)=0$.*

Proof. If there was a positive solution with the above characteristics, then it must be nondecreasing. Extending the function $y(x)$ such that $y(0)=0$, we derive that the function $y(x)$ is continuous on the segment $[0,x_1]$ and differentiable on the interval $(0,x_1)$ for some $x_1>0$. It follows that for every $x\in(0,x_1)$ the inequality $\frac{y(x)-0}{x-0}=y'(\xi)\leqslant y'(x)$ holds for some $\xi\in(0,x)$. Since $y'(x)$ is an increasing function, follows that $y(x)\leqslant xy'(x)\leqslant xy'(x_1)$. From differential equation (2.1) and from above line, it follows that

$$y''(x)=x^a\big(y(x)\big)^{\sigma}\geqslant x^{a+\sigma}\big(y'(x_1)\big)^{\sigma},$$

for each $x\in(0,x_1)$. From the integration we obtain

$$y'(x_1)\geqslant y'(x_1)-y'(x)=\int_x^{x_1} y''(t)dt\geqslant\big(y'(x_1)\big)^{\sigma}\int_x^{x_1} t^{a+\sigma}dt,$$

i.e. $\left(y'(x_1)\right)^{1-\sigma} \geqslant \int_x^{x_1} t^{a+\sigma}\,dt$, which is impossible, since $a+\sigma \leqslant -1$, so that $\int_x^{x_1} t^{a+\sigma}\,dt \to \infty$ when $x \to 0+$. □

If $\sigma < 0$ and $a+\sigma+1 > 0$, then there are infinitely many solutions $y(x)$ of differential equation (2.1), such that $\lim\limits_{x\to 0+} y(x) = 0$, i.e. there are infinitely many solutions such that the graph of the solution "tends to the point $(0,0)$". This result is from [4].

Example 2. Results of Lemma 1 and Theorem 1 can not be extended for every $a > -2$ and $a+\sigma \leqslant -1$. For example, for $a=-1$, $\sigma = 0$ solutions $y(x) = x\ln x + C(x-x_0)$ of the equation $y'' = x^a y^\sigma$ are nondecreasing on the interval (x_0,∞) for every $x_0, C \geqslant 0$. Specially, for $x_0 = 0$ and every $C \geqslant 0$ we have $\lim\limits_{x\to 0+} y(x) = 0$.

Finally, at the end of this section we give conditions on parameters a and σ which assure that the positive solutions of (2.1) have no vertical asymptote at $x=0$.

Theorem 2. *If $\sigma < 0$, $a > -2$ and $y(x)$ is positive solution of differential equation (2.1) defined on the interval $(0,x_0]$, such that $y'(x_0) \leqslant 0$, then*

$$\lim_{x\to +0} y(x) < +\infty .$$

Proof. It follows from equation (2.1) that $y''(x) > 0$ for $x \in [x_1,x_2]$, $x_2 > x_1 > 0$, i.e. $y(x)$ is a convex function. From this and from the condition $y'(x_0) \leqslant 0$, we can conclude that $y'(x) < 0$ for $x \in (0,x_0)$ and $y(x) \geqslant y(x_0)$ for $x \in (0,x_0]$. This, together with (2.1) implies the inequality

$$y''(x) = x^a y^\sigma(x) \leqslant x^a y^\sigma(x_0)\,. \tag{2.2}$$

First consider the case $a \neq -1$. From the inequality (2.2), after integration, we have

$$y'(x_0) - y'(x) = \int_x^{x_0} y''(t)\,dt \leqslant \int_x^{x_0} t^a y^\sigma(x_0)\,dt = \frac{y^\sigma(x_0)}{a+1}\cdot\left(x_0^{a+1} - x^{a+1}\right),$$

for $x \in (0,x_0]$, i.e. after another integration we have

$$-y(x_0) + y(x) \leqslant \frac{y^\sigma(x_0)}{a+1}\left(x_0^{a+1}(x_0-x) + \frac{x_0^{a+2} - x^{a+2}}{a+2}\right) - y'(x_0)(x_0-x)\,.$$

Therefore

$$y(x) \leqslant \frac{y^\sigma(x_0)}{a+1}\left(x_0^{a+1}(x_0-x) + \frac{x_0^{a+2} - x^{a+2}}{a+2}\right) - y'(x_0)(x_0-x) + y(x_0)\,.$$

Since $a > -2$ we conclude that $\lim\limits_{x\to +0} y(x) < +\infty$.

Now, consider the case $a = -1$. Similarly, from the inequality (2.2) we have

$$y'(x_0) - y'(x) \leqslant y^\sigma(x_0)\cdot(\ln x_0 - \ln x),\quad x \in (0,x_0],$$

$$\text{i.e.}\quad y(x) \leqslant y^\sigma(x_0)\cdot(x_0-x)\cdot(\ln x_0 - \ln\bar{x}) - y'(x_0)(x_0-x) + y(x_0)\,,$$

for some $\bar{x} \in (x, x_0)$. Therefore, we conclude that $\lim_{x\to+0} y(x) < +\infty$. □

In [3] it was shown that if $a \leqslant -2$, the result of the previous theorem can not be applied. Also in that paper asymptotic behavior near zero is obtained of each positive solution of equation (2.1) for $a \leqslant -2$.

3. Cauchy problem for an Emden-Fowler equation

Consider the Cauchy problem

$$y'' - x^a y^\sigma = 0,\ y(0) = c,\ y'(0) = \lambda\,, \tag{3.1}$$

where $\sigma < 0$, $a \in \mathbb{R}$ and λ, c are arbitrary constants such that $c > 0$.

First we must precise what is the meaning of Cauchy problem in this situation. Emden-Fowler equation is defined for $x > 0$, but, for example, under assumptions of Lemma 1 and Theorem 2 it follows that $\lim_{x\to 0+} y(x) = c$ exists in $[0,\infty)$, so we will write this shortly as $y(0) = c$. Similarly, since $y''(x) > 0$ for $x > 0$, we have that $\lim_{x\to 0+} y'(x) = \lambda$ exists in $\mathbb{R}\cup\{-\infty\}$ and we will consider situation when $\lambda \in \mathbb{R}$.

The next theorem provides us the existence and uniqueness of solution of the Cauchy problem (3.1).

Theorem 3. *If $\sigma < 0$ and $a > -1$, then Cauchy problem (3.1) has a unique solution defined on some interval $(0,h]$, $h \leqslant 1$ (solution is defined on $(0,h]$ and solution can continuously differentiable extend to 0, such that $y(0) = c > 0$, $y'(0) = \lambda$), for h small enough.*

Proof. First consider the case $\lambda > 0$. Instead of Cauchy problem (3.1) we consider the integral equation

$$y(x) = \lambda x + c + \int_0^x (x-t)t^a y^\sigma(t)\,dt\,. \tag{3.2}$$

The sequence of functions is defined as follows:

$$y_0(x) = \lambda x + c,\ y_{n+1}(x) = \lambda x + c + \int_0^x (x-t)t^a y_n^\sigma(t)\,dt\,. \tag{3.3}$$

Let us show that all functions $y_n(x)$ satisfy the inequalities

$$c + \lambda x \leqslant y_n(x) \leqslant c + 2\lambda x,\ \ x \in [0,h],\ \ n \in \mathbb{N}_0\,, \tag{3.4}$$

for some h independent of n. We prove this assertion by induction. If $n = 0$ then $c + \lambda x = y_0(x) \leqslant 2\lambda x + c$, for $x \in [0,h]$. Let inequalities (3.4) hold, then

$$(c+\lambda x)^\sigma \geqslant y_n^\sigma(x) \geqslant (2\lambda x + c)^\sigma\,,$$

for $x \in [0,h]$. Then

$$y_{n+1}(x) \leqslant \lambda x + c + \int_0^x (x-t)t^a (c+\lambda t)^\sigma\,dt$$

$$\leqslant \lambda x + c + x\int_0^h t^a (c+\lambda t)^\sigma dt \leqslant 2\lambda x + c\,,$$

where h is sufficiently small. Since

$$\lambda x + c \leqslant \lambda x + c + \int_0^x (x-t)t^a y_n^\sigma(t)dt = y_{n+1}(x)\,,$$

so (3.4) holds.

We will show uniform convergence of the sequence $y_n(x)$ on $[0,h]$. Consider the function series $y_0(x) + \sum_{n=1}^{\infty}(y_n(x) - y_{n-1}(x))$. The sequence of partial sums of that series is $S_n(x) = y_0(x) + (y_1(x) - y_0(x)) + \ldots + (y_n(x) - y_{n-1}(x)) = y_n(x)$, i.e. uniform convergence of functional sequence $y_n(x)$ is equivalent to uniform convergence of that function series.

Let us show that

$$|y_n(x) - y_{n-1}(x)| \leqslant x^n\lambda^n,\ \ x\in[0,h],\ \ n\in\mathbb{N}\,, \tag{3.5}$$

for some h independent of n. We prove this assertion by induction. For $n=1$, we have

$$|y_1(x) - y_0(x)| \leqslant x\int_0^x t^a(\lambda t + c)^\sigma dt \leqslant x\int_0^h t^a(\lambda t + c)^\sigma dt \leqslant \lambda x\,,$$

where h is sufficiently small. Let inequality (3.5) hold, then

$$\begin{aligned}|y_{n+1}(x) - y_n(x)| &\leqslant x\int_0^x t^a|y_n^\sigma(t) - y_{n-1}^\sigma(t)|dt\\ &= x|\sigma|\int_0^x t^a|y_n(t) - y_{n-1}(t)|\xi^{\sigma-1}(t)dt,\end{aligned}$$

where $\min\limits_{x\in[0,h]}(y_n(x), y_{n-1}(x)) \leqslant \xi(x) \leqslant \max\limits_{x\in[0,h]}(y_n(x), y_{n-1}(x))$, but

$$(c+\lambda x)^{\sigma-1} \geqslant \xi^{\sigma-1}(x) \geqslant (2\lambda x + c)^{\sigma-1}\,,$$

so we have

$$\begin{aligned}|y_{n+1}(x) - y_n(x)| &\leqslant x|\sigma|\int_0^x t^a(c+\lambda t)^{\sigma-1}x^n\lambda^n dt\\ &\leqslant x^{n+1}\lambda^n|\sigma|\int_0^h t^a(c+\lambda t)^{\sigma-1}dt\,,\end{aligned}$$

i.e. $\quad |y_{n+1}(x) - y_n(x)| \leqslant x^{n+1}\lambda^{n+1}\,,$

where h is sufficiently small. We conclude that

$$|y_n(x) - y_{n-1}(x)| \leqslant x^n\lambda^n \leqslant (\lambda h)^n,\ \ x\in[0,h],\ \ n\in\mathbb{N}\,.$$

From lines above we can conclude that sequence $y_n(x)$ uniformly converges to some function $y(x)$.

Let $y(x) = \lim_{n\to+\infty} y_n(x),\ x \in [0,h]$. Then $y(0) = c$ and $y'(0) = \lambda$. Since we have

$$|y_n^\sigma(x) - y^\sigma(x)| \leqslant \xi^{\sigma-1}(x)\cdot|\sigma|\cdot|y_n(x) - y(x)|,$$

where $\min_{x\in[0,h]}(y_n(x), y(x)) \leqslant \xi(x) \leqslant \max_{x\in[0,h]}(y_n(x), y(x))$, sequence $y_n^\sigma(x)$ uniformly converges to function $y^\sigma(x)$ on $[0,h]$. Because of that we have

$$\lim_{n\to+\infty}\int_0^x (x-t)t^a y_n^\sigma(t)dt = \int_0^x \lim_{n\to+\infty}((x-t)t^a y_n^\sigma(t))dt = \int_0^x (x-t)t^a y^\sigma(t)dt\,.$$

From this we conclude that sequence $y_n(x)$ uniformly converges to function given in (3.2).

Let us prove the uniqueness. Let $y_1(x)$ and $y_2(x)$ be solutions of Cauchy problem (3.1). Then

$$|y_1(x) - y_2(x)| \leqslant x|\sigma|\int_0^x t^a|y_1(t) - y_2(t)|\xi^{\sigma-1}(t)dt\,,$$

where $\min_{x\in[0,h]}(y_1(x), y_2(x)) \leqslant \xi(x) \leqslant \max_{x\in[0,h]}(y_1(x), y_2(x))$, so

$$y_1(x) \geqslant \lambda x + c,\ \ y_2(x) \geqslant \lambda x + c,\ \ x \in [0,h]\,,$$

because $y_1''(x) > 0$ and $y_2''(x) > 0$, for $x \in [0,h]$, we can conclude that

$$y_1^{\sigma-1}(x) \leqslant (\lambda x + c)^{\sigma-1},\ \ y_2^{\sigma-1}(x) \leqslant (\lambda x + c)^{\sigma-1},\ \ x \in [0,h]\,,$$
$$\text{i.e.}\quad \xi^{\sigma-1}(x) \leqslant (\lambda x + c)^{\sigma-1},\ \ x \in [0,h]\,.$$

Finally, we have

$$|y_1(x) - y_2(x)| \leqslant x\cdot\max_{x\in[0,h]}|y_1(x) - y_2(x)|\cdot|\sigma|\int_0^h t^a(\lambda t + c)^{\sigma-1}dt\,,$$
$$\text{i.e.}\quad |y_1(x) - y_2(x)| \leqslant \frac{1}{2}\max_{x\in[0,h]}|y_1(x) - y_2(x)|,\ \ x \in [0,h]\,,$$

where h is sufficiently small. It follows that $y_1 \equiv y_2$.

Now, let us consider the case $\lambda \leqslant 0$. Instead of Cauchy problem (3.1) we consider the integral equation (3.2), where $x \in [0,h]$, for h sufficiently small ($h\lambda + c > 0$). The sequence of functions $y_n(x)$ is defined as (3.3). Let us show that all functions $y_n(x)$ satisfy the inequalities

$$c + \lambda x \leqslant y_n(x) \leqslant c + x,\ \ x \in [0,h],\ \ n \in \mathbb{N}_0\,, \tag{3.6}$$

for some h independent of n. We prove this assertion by induction. If $n = 0$ then $c + \lambda x = y_0(x) \leqslant x + c$, for $x \in [0,h]$. Let inequalities (3.6) hold, then

$$(c + \lambda x)^\sigma \geqslant y_n^\sigma(x) \geqslant (x + c)^\sigma\,,$$

for $x \in [0,h]$. Then

$$y_{n+1}(x) \leqslant c + \int_0^x (x-t)t^a(c+\lambda t)^\sigma dt \leqslant c + x\int_0^h t^a(c+\lambda t)^\sigma dt \leqslant x+c\,,$$

where h is sufficiently small. How it is

$$\lambda x + c \leqslant \lambda x + c + \int_0^x (x-t)t^a y_n^\sigma(t)dt = y_{n+1}(x)\,,$$

then (3.6) holds.

We will show uniform convergence of the sequence $y_n(x)$ on $[0,h]$. For proof of uniform convergence of functional sequence $y_n(x)$ is enough to prove uniform convergence of function series $y_0(x) + \sum_{n=1}^{\infty}(y_n(x) - y_{n-1}(x))$.

Let us show that

$$|y_n(x) - y_{n-1}(x)| \leqslant x^n,\ x \in [0,h],\ n \in \mathbb{N}\,, \tag{3.7}$$

for some h independent of n. We prove this assertion by induction. For $n = 1$, we have

$$|y_1(x) - y_0(x)| \leqslant x\int_0^x t^a(\lambda t + c)^\sigma dt \leqslant x\int_0^h t^a(\lambda t + c)^\sigma dt \leqslant x\,,$$

where h is sufficiently small. Let inequality (3.7) hold, then

$$\begin{aligned} |y_{n+1}(x) - y_n(x)| &\leqslant x\int_0^x t^a|y_n^\sigma(t) - y_{n-1}^\sigma(t)|dt \\ &= x|\sigma|\int_0^x t^a|y_n(t) - y_{n-1}(t)|\xi^{\sigma-1}(t)dt, \end{aligned}$$

where $\min\limits_{x\in[0,h]}(y_n(x), y_{n-1}(x)) \leqslant \xi(x) \leqslant \max\limits_{x\in[0,h]}(y_n(x), y_{n-1}(x))$, so

$$(c+\lambda x)^{\sigma-1} \geqslant \xi^{\sigma-1}(x) \geqslant (x+c)^{\sigma-1}\,,$$

we have

$$|y_{n+1}(x) - y_n(x)| \leqslant x|\sigma|\int_0^x t^a(c+\lambda t)^{\sigma-1}x^n dt \leqslant x^{n+1}|\sigma|\int_0^h t^a(c+\lambda t)^{\sigma-1}dt\,,$$

$$\text{i.e.} \qquad |y_{n+1}(x) - y_n(x)| \leqslant x^{n+1}\,,$$

where h is sufficiently small. We conclude that

$$|y_n(x) - y_{n-1}(x)| \leqslant x^n \leqslant h^n,\ x \in [0,h],\ n \in \mathbb{N}\,.$$

From lines above we can conclude that sequence $y_n(x)$ uniformly converges to some function $y(x)$.

In the same way as $\lambda > 0$, it can be shown that sequence $y_n^\sigma(x)$ uniformly converges to function $y^\sigma(x)$ on $[0,h]$. Because of that we have

$$\lim_{n\to+\infty}\int_0^x (x-t)t^a y_n^\sigma(t)dt = \int_0^x \lim_{n\to+\infty}((x-t)t^a y_n^\sigma(t))dt = \int_0^x (x-t)t^a y^\sigma(t)dt.$$

From this we conclude that sequence $y_n(x)$ uniformly converges to function

$$y(x) = \lambda x + c + \int_0^x (x-t)t^a y^\sigma(t)dt.$$

The proof of uniqueness in this case is the same as the proof when $\lambda > 0$. It follows that $y_1 \equiv y_2$. □

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